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# ON POSITIVE CONTRACTIONS IN $L^p$-SPACES

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ABSTRACT. Let $T$ denote a positive contraction ($T > 0$, $\|T\| < 1$) on a space $L^p(\mu)$ ($1 < p < +\infty$). A primitive $n$th root of unity $\varepsilon$ is in the point spectrum $P\sigma(T)$ iff it is in $P\sigma(T')$; if so, the unimodular group generated by $\varepsilon$ is in both $P\sigma(T)$ and $P\sigma(T')$. In turn, this is equivalent to the existence of $n$-dimensional Riesz subspaces of $L^p$ and $L^q$ ($p^{-1} + q^{-1} = 1$) which are in canonical duality and on which $T$ (resp., $T'$) acts as an isometry. If, in addition, $T$ is quasi-compact then the spectral projection associated with the unimodular spectrum of $T$ (resp., $T'$) is a positive contraction onto a Riesz subspace of $L^p$ (resp., $L^q$) on which $T$ (resp., $T'$) acts as an isometry.

**Introduction.** Since the discovery of the theorems of Perron and Frobenius early in this century, there has been a growing awareness of the role that positivity of an operator plays in its spectral theory. As far as (linear) operators on concrete or abstract Banach lattices are concerned, it has been well known for some time that positivity of an operator is strongly reflected in the properties of its "peripheral" spectrum (i.e., the intersection of $\sigma(T)$ with the spectral circle). For a sketch of the more recent history of the subject, we refer to the bibliographical notes in **[3**, Chapter V**]**; cf. also **[4]**, **[5]**.

This paper is concerned with the presence of roots of unity in the point spectrum $P\sigma(T)$ when $T$ is a positive contraction on a space $L^p(\mu)$ ($1 < p < +\infty$). The ensuing symmetry between $T$ and $T'$ as well as the induced isometries are reminiscent of normal operators in Hilbert space (Theorems 1 and 2). Extensions to spaces $L^1(\mu)$ and $L^\infty(\mu)$ are pointed out, and only slightly weaker versions of Theorems 1 and 2 are valid in any reflexive Banach lattice whose norm and dual norm are strictly monotone.

**1. Positive contractions in $L^p$.** Let $L^p$ denote the complex space $L^p(X, \Sigma, \mu)$ for an arbitrary (positive) measure space $(X, \Sigma, \mu)$ in the sense of **[1]**, **[3]**. *A Riesz subspace* (*vector sublattice*) $F$ is a linear, conjugation invariant subspace of $L^p$ such that $f \in F$ implies $|f| \in F$. It is well known that a Riesz subspace of finite dimension $n$ is linearly and order isomorphic to $\mathbf{C}^n$. If $\dim F = n$, by the *canonical basis* of $F$ we understand the set $\{x_1, \dots, x_n\}$ of generators of extreme rays of $F_+$ normalized to satisfy $\|x_\nu\| = 1$ ($\nu = 1, \dots, n$); this basis is unique (except for numeration). Suppose $F$ and $G$ are $n$-dimensional Riesz subspaces of $L^p$ and $L^q$, respectively ($p^{-1} + q^{-1} = 1$); we say that $F$ and $G$ are in *canonical duality* if $\langle x_\mu, y_\nu \rangle = \delta_{\mu\nu}$ for a suitable numbering of their

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respective canonical bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ ($\langle x, y\rangle$ denoting the standard duality of $L^p$ and $L^q$). The main purpose of this section is to prove this result.

THEOREM 1. *Let $T \geqslant 0$, $\|T\| = 1$, be an operator on $L^p$ $(1 < p < +\infty)$. The following assertions are equivalent*:

(a) *There exists a primitive nth root of unity in the point spectrum of $T$.*

(b) *Every nth root of unity is in the point spectrum of $T$.*

(a′) *Assertion* (a) *for the transpose $T'$ of $T$.*

(b′) *Assertion* (b) *for the transpose $T'$ of $T$.*

(c) *There exist n-dimensional Riesz subspaces $F \subset L^p$, $G \subset L^q$ $(p^{-1} + q^{-1} = 1)$ which are in canonical duality and whose canonical bases are cyclically permuted by $T$ and $T'$, respectively.*

The proof of this theorem is based on several lemmas and propositions phrased in the language of abstract Banach lattices. These will show that a more general result can be derived, but we feel that the spectral behavior of positive contractions is much better illustrated in the concrete case of $L^p$-spaces, where a number of technical hypotheses are automatically satisfied.

Specifically, we shall investigate the relationship of the following assertions, where $E$ denotes a complex Banach lattice [3, II, §11] and $T$: $E \to E$ a positive linear operator. $P\sigma(T)$ will denote the point spectrum of $T$.

($\alpha$) *There exists a primitive nth root of unity in $P\sigma(T)$.*

($\beta$) *There exist vectors $x > 0$ such that $T^n x = x$ and $x, Tx, \ldots, T^{n-1}x$ are pairwise orthogonal.*

($\gamma$) *There exist orthogonal positive vectors $x_0, \ldots, x_{n-1}$ in $E$ and orthogonal positive forms $\varphi_0, \ldots, \varphi_{n-1}$ in $E'$ such that*

$$Tx_{\nu+1} = x_\nu, \qquad T'\varphi_\nu = \varphi_{\nu+1}\ (\nu \bmod n) \quad \textit{and} \quad \langle x_\mu, \varphi_\nu\rangle = \delta_{\mu\nu}.$$

It is easy to prove the implications $(\gamma) \Rightarrow (\beta) \Rightarrow (\alpha)$. For the proof of the converse implications, these abbreviations will be useful:

($*$) *For all $x \in E_+$, $x \leqslant Tx$ implies $x = Tx$.*

($*$)′ *For all $\varphi \in E'_+$, $\varphi \leqslant T'\varphi$ implies $\varphi = T'\varphi$.*

(G) $(\lambda - 1)(\lambda - T)^{-1}$ *is uniformly bounded for $\lambda > 1$.*

Condition (G), a growth condition for the resolvent of $T$, plays an important role in the spectral theory of positive operators (cf. [3, V. 4]); we shall also have need for this extended version of the lemma on unimodular eigenfunctions ([3, V. 4.2]).

Let $K$ denote a compact space and let $T$ be a Markov operator on $C(K)$ (i.e., let $T \geqslant 0$ and $Te = e$ for the constant-one function $e$ on $K$). If $\varepsilon f = Tf$, where $f \in C(K)$, $|f| = e$, and $|\varepsilon| = 1$, then $f$ is called a *unimodular eigenfunction of $T$ and $\varepsilon$ a unimodular eigenvalue.*

LEMMA. *The unimodular eigenfunctions of $T$ form a group $G$ under pointwise multiplication. The map $f \mapsto \varepsilon$ is a character of $G$ (and the group $G^*$ of unimodular eigenvalues isomorphic to $G/G_1$, $G_1$ the subgroup fixed under $T$).*

*Moreover, if $\varepsilon \in G^*$ is a primitive nth root of unity, there exist functions $h > 0$ in $C(K)$ such that $T^n h = h$ and $h, Th, \ldots, T^{n-1}h$ are mutually orthogonal.*

PROOF. The action of $T$ on $C(K)$ can be expressed by $Tf(s) = \int f(t)\, d\mu_s(t)$, where $\mu_s := T'\delta_s$ ($\delta_s$ the Dirac measure at $s \in K$) is a probability measure, because $T$ is Markov, and where $s \mapsto \mu_s$ is $w^*$-continuous. Now if $\varepsilon f(s) = \int f(t)\, d\mu_s(t)$ where $|f| = e$ then, since $\mu_s$ is a probability measure, we must have $f(t) = \varepsilon f(s)$ for all $t$ in the (closed) support of the Radon measure $\mu_s$. This proves the first part of the lemma (see [3, V.4.2]).

Now let $G_\varepsilon$ denote the group of unimodular eigenfunctions belonging to one of the eigenvalues $1, \varepsilon, \ldots, \varepsilon^{n-1}$. ($G_\varepsilon$ is the inverse image of $\Gamma_\varepsilon$, the group of $n$th roots of unity, under the homomorphism $G \to G^*$ given by $f \mapsto \varepsilon$.) The linear span $F$ of $G_\varepsilon$ in $C(K)$ is a conjugation invariant subalgebra $F$ containing $e$, and $T^n|F$ is the identity; the same is obviously true of the closure $\bar{F}$ in $C(K)$. Hence by the complex Stone-Weierstrass theorem, $\bar{F} \cong C(K_0)$ where $K_0$ is a compact quotient of $K$. Now let $f \in G_\varepsilon$, $\varepsilon f = Tf$. We define

$$M_\nu = \{t \in K_0 \colon 2\pi\nu/n \leqslant \arg f(t) < 2\pi(\nu + 1)/n\} \qquad (\nu = 0, \ldots, n-1).$$

Then $K_0 = \bigcup_0^{n-1} M_\nu$ and $s \in M_\nu$ implies support $\mu_s \subset M_{\nu+1}$ ($\nu$ mod $n$). On the other hand, at least one of the $M_\nu$, $M_1$ say, must have nonvoid interior in $K_0$. If $h > 0$ is a function in $C(K_0)$ vanishing outside $M_1$, then $(Th)(s) = 0$ for all $s \notin M_0$. By iteration it follows that $h, Th, \ldots, T^{n-1}h$ are orthogonal and, clearly, $T^n h = h$.

PROPOSITION 1. *If $T$ satisfies $(*)$ then $(\alpha) \Rightarrow (\beta)$.*

PROOF. Let $\varepsilon x = Tx$, where $x \neq 0$ and $\varepsilon$ is a primitive $n$th root of unity. Then $|x| = |\varepsilon x| = |Tx| \leqslant T|x|$, since $T$ is positive, and hence $|x| = T|x|$ by $(*)$. Denote by $\mathfrak{U}$ the (complex) interval $\{z \colon |z| \leqslant |x|\}$ in $E$. Under the norm whose unit ball is $\mathfrak{U}$, the principal ideal $E_{|x|} := \bigcup_1^\infty n\mathfrak{U}$ of $E$ is isometrically order isomorphic to $C(K)$ (complex) for some compact space $K$ (Kakutani's theorem). Identifying $E_{|x|}$ with $C(K)$, we note that $T$ induces a Markov operator on $C(K)$ for which $x$ constitutes a unimodular eigenfunction with eigenvalue $\varepsilon$. The remainder now follows from the preceding lemma; this proves $(\beta)$.

PROPOSITION 2. *Suppose that $E$ has order continuous norm, that $T$ satisfies* (G) *and $T'$ satisfies $(*)'$. Then $(\beta) \Rightarrow (\gamma)$.*

PROOF. Let $x > 0$ be a vector as specified in $(\beta)$; defining $x_\nu$ ($\nu = 0, 1, \ldots, n-1$) by $x_\nu = T^{n-\nu}x$, we obtain $Tx_{\nu+1} = x_\nu$ ($\nu$ mod $n$) and $x_0, \ldots, x_{n-1}$ are pairwise orthogonal. Denoting by $J_\nu$ the closed ideal of $E$ generated by $x_\nu$ we have $T(J_{\nu+1}) \subset J_\nu$. Since $E$ has order continuous norm, each $J_\nu$ is a projection band [3, II.5.14]; let $P_\nu \colon E \to J_\nu$ denote the corresponding band projection. Letting $e = \sum_{\nu=0}^{n-1} x_\nu$ we have $Te = e$; since $T$ satisfies (G), there exists a positive linear functional $\varphi \in E'$ satisfying $\varphi = T'\varphi$ and

$\varphi(e) > 0$ ([3, V.4.8]; note that condition (G) as given above implies the spectral radius of $T$ to be $\leqslant 1$).

We now define $\varphi_\nu = \varphi \circ P_\nu$ ($\nu = 0, 1, \ldots, n-1$). We have $0 \leqslant P \leqslant I$ for every band projection; additionally, since $T(J_{\nu+1}) \subset J_\nu$, we obtain $P_\nu T P_{\nu+1} = TP_{\nu+1}$. Hence,

$$T'\varphi_\nu = \varphi \circ P_\nu T \geqslant \varphi \circ P_\nu T P_{\nu+1} = \varphi \circ TP_{\nu+1} = \varphi \circ P_{\nu+1} = \varphi_{\nu+1}$$

for all $\nu$ ($\nu \bmod n$). Letting $\psi = \Sigma_0^{n-1} \varphi_\nu$ we obtain $T'\psi = \Sigma_0^{n-1} T'\varphi_\nu \geqslant \Sigma_0^{n-1} \varphi_{\nu+1} = \psi$, but $T'\psi \geqslant \psi$ implies $T'\psi = \psi$ by condition $(*)'$. This means

$$\sum_{\nu=0}^{n-1} (T'\varphi_\nu - \varphi_{\nu+1}) = 0;$$

since in this sum all summands are $\geqslant 0$, it follows that $T'\varphi_\nu = \varphi_{\nu+1}$ ($\nu \bmod n$). Obviously $\varphi_\nu(x_\mu) = 0$ whenever $\nu \neq \mu$. Further, $\varphi_\nu(x_\nu) = \varphi(P_\nu x_\nu) = \varphi(x_\nu)$ and $\varphi(x_\nu) = \varphi(Tx_{\nu+1}) = \varphi(x_{\nu+1})$ because $T'\varphi = \varphi$. Thus the value of $\varphi_\nu(x_\nu)$ is independent of $\nu$; if we normalize the functional $\varphi$ by requiring that $\varphi(e) = n$, we obtain $\varphi_\nu(x_\nu) = 1$ for all $\nu$ and hence, $\varphi_\nu(x_\mu) = \delta_{\nu\mu}$. This proves ($\gamma$), since the $\varphi_\nu$ are clearly orthogonal.

Proof of Theorem 1. If $E = L^p$ and $E' = L^q$, where $1 < p,q < +\infty$ and $p^{-1} + q^{-1} = 1$, every positive contraction $T$ on $E$ satisfies (G) and $(*)$; similarly, its transpose $T'$ satisfies (G) and $(*)'$ (immediate verification). Thus by Propositions 1 and 2, the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold if the orthogonal sets $\{x_\nu\}$ and $\{\varphi_\nu\}$ ($\nu = 0, \ldots, n-1$) of ($\gamma$) can be chosen to satisfy $\|x_\nu\| = \|\varphi_\nu\| = 1$ for all $\nu$. The equivalence of all statements of Theorem 1 follows from the trivial implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) and the symmetry between $T$ and $T'$.

The remainder of the proof thus consists in a metric refinement of Proposition 2. Since $Tx_{\nu+1} = x_\nu$ ($\nu \bmod n$) for all $\nu$ and $\|T\| = 1$, the $x_\nu$ must have equal norms which can be chosen to be 1. It then follows that $\|e\|^p = \|\Sigma x_\nu\|^p = \Sigma \|x_\nu\|^p = n$. If we can choose the invariant functional $\varphi = T'\varphi > 0$ in such a way that $\varphi(e) = n$ and $\|\varphi\|^q \leqslant n$ (this will be shown to be possible at once) then, since $\varphi \geqslant \Sigma_0^{n-1} \varphi_\nu$ and the $\|\varphi_\nu\| =: k$ are equal ($T'$ being a contraction), we must have $n \geqslant \|\varphi\|^q \geqslant \Sigma_0^{n-1} \|\varphi_\nu\|^q = nk^q$ and hence, $k^q \leqslant 1$ and $k \leqslant 1$; because of $\varphi_\nu(x_\nu) = 1$, it follows that $k = 1$. (In turn, this implies that $\varphi = \Sigma \varphi_\nu$.)

To construct $\varphi$ with the desired properties we proceed as follows. Since $\|e\|^q = n$, the linear functional $\psi_0$ on the one-dimensional Riesz subspace $\{\xi e: \xi \in \mathbf{C}\}$ of $E$ defined by $\psi_0(e) = n$ has norm $\|\psi_0\| = n^{1/q}$. There exists a norm preserving positive extension $\psi$ of $\psi_0$ to $E$ [3, I.5.6]. Consider the family $\Phi = \{(\lambda - 1)(\lambda - T')^{-1}\psi: \lambda > 1\}$ of positive linear functionals on $E$. Since $Te = e$, an easy computation shows that each $\tau \in \Phi$ satisfies $\tau(e) = n$; moreover $\|\tau\| \leqslant \|\psi\| = n^{1/q}$ for all $\tau$, since $T'$ is a contraction. Thus $\Phi$ is relatively $w^*$-compact; if $\varphi$ is any weak cluster point of $\Phi$ as $\lambda \downarrow 1$, it is easy to see that $\varphi = T'\varphi$, $\varphi(e) = n$, and $\|\varphi\| \leqslant n^{1/q}$.

This completes the proof of Theorem 1.

REMARK. Suppose $(X, \Sigma, \mu)$ is a localizable measure space, so that $L^\infty(\mu)$ is the dual of $L^1(\mu)$, and that $T$ is a positive contraction in $L^1(\mu)$ with transpose $T'$ in $L^\infty(\mu)$. Propositions 1, 2 and the preceding proof show the following to hold:

(i) If $T'$ satisfies $(*)'$, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (b$'$) $\Leftrightarrow$ (a$'$) of Theorem 1 are valid with regard to the duality $\langle L^1(\mu), L^\infty(\mu)\rangle$.

(ii) If, in addition, the fixed space of $T''$ is contained in $L^1(\mu)$ (e.g., if $T$ is quasi-compact, cf. §2 below), then also (b$'$) $\Rightarrow$ (c).

**2. Spectral projections.** We start from the assumption that $T$ is a positive operator on a (complex) Banach lattice $E$ satisfying assertion $(\gamma)$ of §1. If $E_0$ denotes the linear span of $\{x_\nu : \nu = 0, \ldots, n-1\}$ and if $E_1 := \bigcap_{\nu=0}^{n-1} \varphi_\nu^{-1}(0)$ then, clearly, $E = E_0 + E_1$ is a topological direct sum reducing $T$. As before, let $\Gamma$ denote the circle group, $\Gamma_\varepsilon$ the group of $n$th roots of unity ($\varepsilon$ a primitive $n$th root of 1), and $\sigma(T)$ the spectrum of $T$. The terms *spectral set* and *spectral projection* are employed as in [1] (see also below).

PROPOSITION 3. *Suppose $T$, $0 \leq T \in \mathfrak{L}(E)$, satisfies $(\gamma)$ (§1) and that $T|E_1$ has spectral radius $< 1$. Then $\sigma(T) \cap \Gamma = \Gamma_\varepsilon$ and $\Gamma_\varepsilon$ is a spectral set of $T$; the associated spectral projection equals $\sum_{\nu=0}^{n-1} \varphi_\nu \otimes x_\nu$ and hence is positive. Moreover, $k^{-1}(\sum_{\kappa=0}^{k-1} T^\kappa)$ converges in norm to the projection $n^{-1}(\sum_0^{n-1} \varphi_\nu) \otimes (\sum_0^{n-1} x_\nu)$ as $k \to \infty$.*

PROOF. Setting

$$y_k = \frac{1}{\sqrt{n}} \sum_{\nu=0}^{n-1} \varepsilon^{k\nu} x_\nu \quad \text{and} \quad \psi_k = \frac{1}{\sqrt{n}} \sum_{\nu=0}^{n-1} \bar{\varepsilon}^{k\nu} \varphi_\nu$$

$(k = 0, \ldots, n-1)$ and recalling that $Tx_{\nu+1} = x_\nu$, $T'\varphi_\nu = \varphi_{\nu+1}$ ($\nu \bmod n$) it follows that $Ty_k = \varepsilon^k y_k$, $T'\psi_k = \varepsilon^k \psi_k$ for all $k$. Moreover, $\det(\varepsilon^{k\nu})$ is a van der Monde determinant $\neq 0$, and so $\{y_k : k = 0, \ldots, n-1\}$ is a basis of the vector space $E_0$. Since $\varphi_\mu(x_\nu) = \delta_{\mu\nu}$, it follows that $\psi_k(y_l) = \delta_{kl}$ and consequently, the functionals $\psi_k$ restricted to $E_0$ are the coefficient functionals with respect to the basis $\{y_k\}$ of $E_0$. Therefore, the spectrum of $T|E_0$ is precisely $\Gamma_\varepsilon$. The hypothesis $r(T|E_1) < 1$ now ensures that $\Gamma_\varepsilon$ is a spectral set for $T$ (i.e., an open and closed subset of $\sigma(T)$). Since $\det(\bar{\varepsilon}^{k\nu})$ is likewise $\neq 0$, we conclude that $E_1 = \bigcap_{k=0}^{n-1} \psi_k^{-1}(0)$. Therefore, $P = \sum_0^{n-1} \varphi_\nu \otimes x_\nu$ and $\tilde{P} = \sum_0^{n-1} \psi_k \otimes y_k$ are projections with identical kernels and ranges, and hence $P = \tilde{P}$. On the other hand, if $\varepsilon^k$ $(0 \leq k \leq n-1)$ is any element of $\Gamma_\varepsilon$, the spectral projection

$$P_k = (2\pi i)^{-1} \int_{c_k} (\lambda - T)^{-1} \, d\lambda,$$

where $c_k$ is a small, positively oriented circle about $\varepsilon^k$, must equal $\psi_k \otimes y_k$ (again because $\psi_k \otimes y_k$ has identical kernel and range as $P_k$). Thus, since the spectral projection associated with $\Gamma_\varepsilon$ is $\sum_{k=0}^{n-1} P_k$ it necessarily equals $P$.

Finally, it is clear that $T$ is uniformly ergodic with $k^{-1}(\sum_{\kappa=0}^{k-1} T^\kappa)$ norm converging to $P_0 = \psi_0 \otimes y_0$ for $k \to \infty$; by definition of $\psi_0$ and $y_0$, this is the assertion.

To relate Proposition 3 to our previous results (especially Theorem 1), some restriction on $\sigma(T) \cap \Gamma$ has to be imposed. In the present setting, the appropriate concept appears to be quasi-compactness of $T$: If $T$ is a (bounded, linear) operator on a Banach space $X$, $T$ is called *quasi-compact* if $\|T^m - K\| < 1$ for some $m \in \mathbf{N}$ and some compact operator $K \in \mathfrak{L}(X)$. If the spectral radius $r(T)$ is $\leqslant 1$, this amounts to requiring that $\sigma(T) \cap \Gamma$ be a finite spectral set and that the associated spectral projection be of finite rank. If $T$ is a positive operator on a Banach lattice $E$, satisfying $\|T^n\|/n \to 0$ for $n \to \infty$, Lin [2] has shown that $T$ is quasi-compact iff $T$ is uniformly ergodic with finite dimensonal fixed space.

THEOREM 2. *Let $T$ be a positive contraction on $E = L^p$ $(1 < p < +\infty)$ which is quasi-compact. The spectral projection $P$ associated with $\sigma(T) \cap \Gamma$ is a positive contraction, and $E_0 = PE$ is a finite dimensional Riesz subspace of $E$ on which $T$ acts as an isometry permuting the canonical basis.*

*An analogous result holds for the transpose $T'$ of $T$ on $E' = L^q$ $(p^{-1} + q^{-1} = 1)$, and the Riesz subspaces $PE$ and $P'E'$ are in canonical duality.*

PROOF. Quasi-compactness of $T$ implies the fixed space $F := \{x: Tx = x\}$ to be finite dimensional; let $m = \dim F$. We prove the theorem first for the special case $m = 1$, and we may and shall suppose that $F \neq \{0\}$.

If $\varepsilon$ is any unimodular eigenvalue of $T$ with eigenvector $x$, then $|x| = T|x|$ by condition $(*)$ and so $|x| \in F$. Since $\dim F = 1$ by our present assumption, all eigenvectors of $T$ pertaining to eigenvalues in $\Gamma$ define, after normalization, unimodular eigenfunctions of the operator $T_0$ induced on $E_{|x|} \cong C(K)$, and conversely. Applying the lemma of §1 shows that the unimodular spectrum of $T$ is a finite subgroup of $\Gamma$ and hence, generated by some root of unity $\varepsilon$; if $n$ denotes the order of this group, then by Theorem 1 we have orthogonal subsets $\{x_0, \dots, x_{n-1}\}$ and $\{\varphi_0, \dots, \varphi_{n-1}\}$ of $L^p$ and $L^q$, respectively, satisfying $Tx_{\nu+1} = x_\nu$, $T'\varphi_\nu = \varphi_{\nu+1}$ ($\nu$ mod $n$) and $\varphi_\mu(x_\nu) = \delta_{\mu\nu}$, $\|x_\nu\| = \|\varphi_\nu\| = 1$. By Proposition 3, the spectral projection associated with $\sigma(T) \cap \Gamma$ is $P = \Sigma_0^{n-1} \varphi_\nu \otimes x_\nu$ and positive; we have to show that $\|P\| = 1$. Since $\{\varphi_\nu\}$ is the canonical basis of the $n$-dimensional Riesz space $P'E'$, the $\varphi_\nu$ are unique except for numeration; thus from the proof of Proposition 2 it follows that $\varphi_\nu = \varphi \circ P_\nu$ for $\varphi = \Sigma_0^{n-1} \varphi_\nu$; this implies $\varphi_\nu = \varphi_\nu \circ P_\nu$ for $\nu = 0, \dots, n-1$. Therefore,

$$\|Px\|^p = \sum_0^{n-1} |\varphi_\nu(x)|^p = \sum_0^{n-1} |\varphi_\nu(P_\nu x)|^p \leqslant \sum_0^{n-1} \|P_\nu x\|^p \leqslant \|x\|^p$$

for all $x \in E = L^p$, since the $P_\nu$ are mutually orthogonal band projections. It follows that $\|P\| = 1$ as asserted. This proves Theorem 2 if $\dim F = m = 1$.

In settling the case $m > 1$, the basic observation is that $F$ is a (complex) Riesz subspace of $E$. In fact, since $T$ satisfies $(*)$ (§1), $x = Tx$ implies

$|x| = T|x|$ (and, clearly, $F$ is conjugation invariant). Now let $\{u_\mu: \mu = 1, \ldots, m\}$ denote the canonical basis of $F$. If $\varepsilon$, $|\varepsilon| = 1$, is an eigenvalue of $T$ and $\varepsilon z = Tz$, then $|z| = \Sigma_{\mu=1}^m \beta_\mu u_\mu$ and so $z = \Sigma_{\mu=1}^m z_\mu$ where $|z_\mu| < \beta_\mu u_\mu$, by the complex version of the decomposition property (see [3, II.11.2]). Since $Tu_\mu = u_\mu$ and the $u_\mu$ are orthogonal, it follows that $\varepsilon z_\mu = Tz_\mu$ for each $\mu = 1, \ldots, m$. Applying the lemma of §1 to each of the principal ideals $E_\mu$ generated by $u_\mu$, respectively, shows that the eigenvalues $\varepsilon \in \sigma(T) \cap \Gamma$ with eigenvectors in $E_\mu$ form a (finite) subgroup $\Gamma_\mu$ of $\Gamma$ of order $n_\mu$, say; clearly, then, $\sigma(T) \cap \Gamma = \bigcup_{\mu=1}^m \Gamma_\mu$. As in the proof of Proposition 1, we construct $m$ sets $\{x_0^{(\mu)}, \ldots, x_{n_\mu-1}^{(\mu)}\} \subset E_\mu$ of normalized, orthogonal vectors, each of which sets is cyclically permuted by $T$; of course, these sets are pairwise orthogonal, since the $E_\mu$ are.

Now we define $e_\mu$ to be the positive scalar multiple of $u_\mu$ for which $\|e_\mu\| = (n_\mu)^{1/p}$ and, repeating the method applied at the end of the proof of Theorem 1, we can find a functional $\varphi \in E'_+$ satisfying $\varphi = T'\varphi$, $\varphi(e_\mu) = n_\mu$ for all $\mu$, and $\|\varphi\| = (\Sigma_1^m n_\mu)^{1/q}$. (Here $\Sigma_1^m n_\mu$ is the total number of unimodular eigenvalues of $T$, counted according to their multiplicity.) The final step consists in constructing the corresponding families $\{\varphi_0^{(\mu)}, \ldots, \varphi_{n_\mu-1}^{(\mu)}\}$ in $E' \cong L^q$: Denoting by $P_\nu^{(\mu)}$ the band projection of $E$ onto the band generated by $x_\nu^{(\mu)}$, we define $\varphi_\nu^{(\mu)} = \varphi \circ P_\nu^{(\mu)}$ for all $\mu,\nu$. As in the proof of Proposition 2, it follows that each of the (orthogonal, normalized) sets $\{\varphi_0^{(\mu)}, \ldots, \varphi_{n_\mu-1}^{(\mu)}\}$ is cyclically permuted by $T'$; a repeated application of Proposition 3 now shows that

$$P = \sum_{\mu=1}^{m} \sum_{\nu=0}^{n_\mu-1} \varphi_\nu^{(\mu)} \otimes x_\nu^{(\mu)}$$

is the spectral projection associated with $\sigma(T) \cap \Gamma$. Hence $P > 0$ and, since $\varphi_\nu^{(\mu)} = \varphi_\nu^{(\mu)} \circ P_\nu^{(\mu)}$ for all $\mu,\nu$, the result that $\|P\| = 1$ can be derived exactly as in the case $m = 1$ above.

COROLLARY 1. *Under the assumptions of Theorem 2, the operator $T$ has a decomposition $T = U + R$ with these properties*: $UR = RU = 0$, $r(R) < 1$, $U > 0$, $\|U\| < 1$ *and* $U^k = P$.

Here, of course, $k$ is the least common multiple of the numbers $n_\mu$ ($\mu = 1, \ldots, m$) introduced above.

COROLLARY 2. *If, under the assumptions of Theorem 2, $E = L^2$ is a Hilbert space, the spectral projection $P$ associated with $\sigma(T) \cap \Gamma$ is hermitian and the operator $T|PE$ is unitary.*

The proof of these corollaries is easy and can be omitted. We remark in conclusion that Theorem 2 extends to positive contractions on $L^1$ or $L^\infty$ under the additional hypotheses given in the remark at the end of §1.

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